HDY-003-1163003 Seat No. ______

M. Sc. (Mathematics) (Sem. III) (CBCS) Examination

November / December – 2017

3003 : Number Theory - I

(Old & New Course)

Faculty Code : 003
Subject Code : 1163003

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) There are five questions.
(2) All questions are compulsory.
(3) Each question carries 14 marks.
(4) Figures to the right indicate full marks.

1 Fill in the blanks : (each question carries 2 marks) **14**

(i) If p is a prime number and n is a positive integer then the number of positive divisors of p^n is ________.

(ii) If p and q are distinct primes then $p^m q^n$ has ________ positive divisors. $(m, n \in \mathbb{N})$

(iii) If p is a prime of the form _____ then $x^2 + 1 \equiv 0 \pmod{p}$ has no solutions.

(iv) If p is a prime number and n a positive integer then the number of positive integers relatively prime to $p^n =$ ________

(v) If $n = 1001 \times 59$ then $\varnothing(n) =$ _________

(vi) If p is a prime number and p does not divide a then $a^{p-1} \equiv 1 \pmod{p}$. This theorem is called ______ theorem.

(vii) If m divides ab then $\frac{m}{(a, m)}$ divides ________

2 Attempt any **two** :

(i) Suppose $p_1, p_2, \ldots\ldots\ldots p_k$ are the first k primes then prove that $p_1, p_2, \ldots\ldots\ldots p_k + 1$ is a prime number and hence deduce that there infinitely many primes. **7**

(ii) State and prove Division Algorithm. **7**

(iii) State and prove Wilson's theorem. **7**

3 All are compulsory :

(i) State and prove Hensel's Lemma. **6**

(ii) Find the smallest positive integer x such that the remainder is 10 when it is divided by 11, the remainder is 12 when it divided by 13 and the remainder is 6 when it is divided by 7. **4**

(iii) Suppose $(a, m) = 1$. Prove that there is a unique integer x in the complete residue system (mod m) such that $ax \equiv 1 \pmod{m}$. **4**

OR

3 All are compulsory :

(i) Suppose $f(x)$ is a polynomial with integer coefficients, p is a prime number and $f(x) \equiv 0 \pmod{p}$ has degree n. Prove that $f(x) \equiv 0 \pmod{p}$ has atmost n solutions in any complete residue system (mod p). **7**

(ii) First find the solutions of $f(x) \equiv 0 \pmod{3}$, $f(x) \equiv 0 \pmod{5}$, $f(x) \equiv 0 \pmod{7}$ and use them to find all solutions of $f(x) \equiv 0 \pmod{105}$. Here $f(x) = x^2 - 1$. **7**

4 Attempt any **two** :

(i) Determine which of the following have primitive roots and if an integer has a primitive root then find atleast two primitive roots : 5, 5^2 , 82, 12 and 35. **7**

(ii) If $\alpha \geq 3$ then prove that the set **7**

$$\left\{5, 5^2, 5^3, ...5^{2\alpha-2}\right\} \cup \left\{-5, -5^2, -5^3, ... -5^{2^{\alpha-2}}\right\}$$ is a

reduced residue system $\left(\text{mod } 2^\alpha\right)$.

(iii) Suppose f is a multiplicative function then prove that the function F defined by $F(n) = \sum_{d/n} f(d)$ is a multiplicative function. **7**

5 Do as directed : (Each carries 2 marks) **14**

(i) Write the statement of mobius inversion formula.

(ii) Find the value of $\varnothing(101)$ using mobius inversion formula.

(iii) Find the highest power of 31 which divides 47321 !

(iv) Find the number of positive divisors of 2016.

(v) Find the values of $w(n)$ for n = 49, 55, 101 × 83, 105.

(vi) Give an example of a multiplicative function which is not totally multiplicative.

(vii) Find $\sigma(n)$ for n = 150, 307.